GRADUALLY VARIED FLOW

INTRODUCTION

What is a gradually varied flow (GVF)?

A steady non-uniform flow in a prismatic channel with gradual changes in water-surface elevation is a gradually varied flow (GVF).

Typical examples of GVF:

(i) the backwater produced by a dam or weir across a river
(ii) the drawdown produced at a sudden drop in a channel

In a GVF, the bed slope of channel, water surface slope and energy slope will all differ from each other. *Why?*

As there is a gradual change in water-surface elevation, there is a change in velocity of flow from section to section along the length of the channel. As a consequence, the bed slope, water surface slope and energy slope differ from each other.

Basic assumptions involved in the analysis of GVF:

1. The pressure distribution at any section is assumed to be hydrostatic. This assumption is based on the definition of the flow to have a gradually varied water surface. As gradual changes in water surface curvature give rise to negligible normal accelerations, the departure from the hydrostatic pressure distribution is negligible.

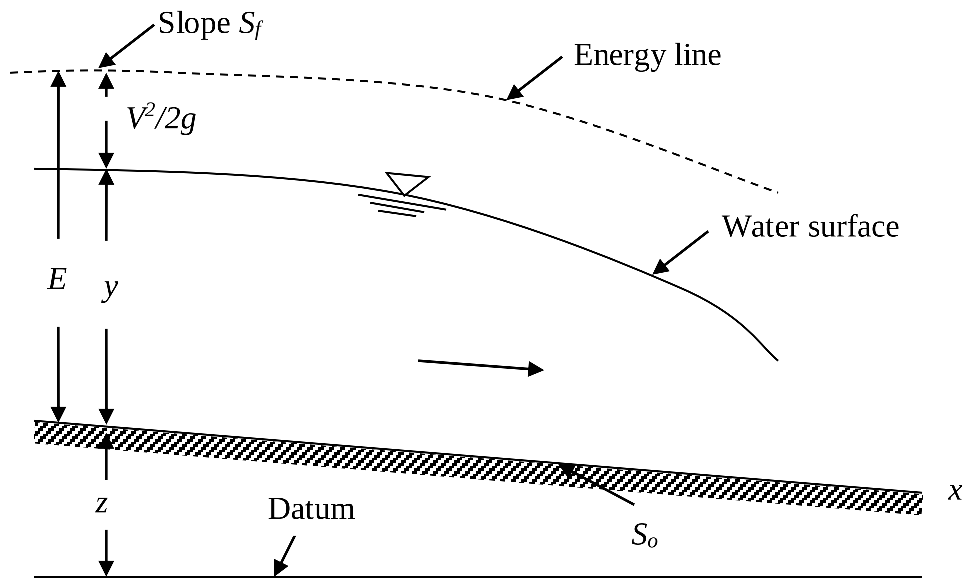
2. the resistance to flow at any depth is assumed to be given by the corresponding uniform flow equation, such as the Manning's equation, with the condition that the slope term to be used in the equation is the energy slope and not the bed slope. Thus, if the depth of flow at any section in a GVF is y, the energy slope S_f is given by

$$S_f = \frac{n^2 V^2}{R^{4/3}}$$

where n is the Manning's roughness coefficient, V is the mean velocity of flow in the channel, R is the hydraulic radius of the channel section where the depth of flow is y.

DIFFERENTIAL EQUATION OF GVF:

Figure shows a schematic sketch of a gradually varied flow.



Schematic sketch of GVF

Let the bed slope S_o of the channel be small. When the channel is straight, prismatic and uniform flow or gradually varied flow take place, the kinetic energy correction factor α can be assumed to be unity (that is, α = 1.0). The total energy H can be expressed as:

$$H = z + E = Z + \left(y + \frac{V^2}{2g} \right) \tag{2}$$

where z = elevation of the channel section above the horizontal datum surface
E = specific energy
y = depth of flow in the section
V = mean velocity of flow at the section

The longitudinal direction (x – direction) is considered along the channel length. As the water surface profile varies along the x – direction, the depth of flow y and total energy H are functions of x.

Differentiating (2) with respect to x,

$$\frac{dH}{dx} = \frac{dz}{dx} + \frac{dE}{dx} = \frac{dz}{dx} + \left\{ \frac{dy}{dx} + \frac{d}{dx}\left(\frac{V^2}{2g} \right) \right\} \qquad (3)$$

In equation (3), what are represented by $\frac{dH}{dx}, \frac{dz}{dx}, \frac{dy}{dx}$ and $\frac{d}{dx}\left(\frac{V^2}{2g} \right)$

$\frac{dH}{dx}$ represents the variation in total energy H along the longitudinal direction (x – direction). It represents the slope of the energy line (S_f). it should be noted that the total energy of the flow always decreases in the direction of flow. Hence,

$$\frac{dH}{dx} = -S_f \qquad (4)$$

$\frac{dz}{dx}$ represents the variation in elevation Z of the channel bed along the longitudinal direction (x – direction). It represents the bottom slope of the channel (S_o). It should be noted that the elevation of the channel bed decreases in the direction of flow. Hence,

$$\frac{dz}{dx} = -S_o \qquad (5)$$

$\frac{dy}{dx}$ represents the variation in depth of flow y along the longitudinal direction (x – direction). It represents the slope of the water – surface with respect to the bottom of the channel.

$\frac{d}{dx}\left(\frac{V^2}{2g} \right)$ represents the variation in kinetic energy of flow in the longitudinal direction.

$$\frac{d}{dx}\left(\frac{V^2}{2g} \right) = \frac{d}{dx}\left(\frac{Q^2}{2gA^2} \right) \qquad \text{(since, } V = \frac{Q}{A}\text{)}$$

$$= \frac{d}{dy}\left(\frac{Q^2}{2gA^2} \right)\frac{dy}{dx}$$

As the wetted area A of flow at any channel section is a function of depth of flow y, the above differential becomes

$$\frac{d}{dy}\left(\frac{Q^2}{2gA^2}\right)\frac{dy}{dx} = -2A^{-3}\left(\frac{Q^2}{2g}\right)\left(\frac{dA}{dy}\right)\frac{dy}{dx} = -\left(\frac{Q^2}{gA^3}\right)\left(\frac{dA}{dy}\right)\frac{dy}{dx}$$

Since, $\frac{dA}{dy} = T$, where T is the width of flow at the water surface at the channel section,

$$\frac{d}{dx}\left(\frac{V^2}{2g}\right) = -\left(\frac{Q^2}{gA^3}\right)T\frac{dy}{dx} = -\left(\frac{Q^2T}{gA^3}\right)\frac{dy}{dx} \qquad (6)$$

Now, substituting the expressions for $\frac{dH}{dx}, \frac{dz}{dx}$ and $\frac{d}{dx}\left(\frac{V^2}{2g}\right)$ from equations (4), (5) and (6) respectively in equation (3),

$$-S_f = -S_o + \frac{dy}{dx} - \left(\frac{Q^2T}{gA^3}\right)\frac{dy}{dx}$$

$$\Rightarrow -S_f = -S_o + \frac{dy}{dx}\left(1 - \frac{Q^2T}{gA^3}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-S_f + S_o}{\left(1 - \frac{Q^2T}{gA^3}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{S_o - S_f}{\left(1 - \frac{Q^2T}{gA^3}\right)} \qquad (7)$$

Equation (7) forms the *basic differential equation of gradually varied flow*. It is also known as the *dynamic equation of GVF*. When the kinetic energy correction factor α is greater than 1.0, equation (7) will be expressed as;

$$\frac{dy}{dx} = \frac{S_o - S_f}{\left(1 - \frac{Q^2T}{gA^3}\right)} = \frac{S_o - S_f}{\left(1 - \frac{\alpha Q^2T}{gA^3}\right)} \qquad (8)$$

What are the other forms in which equation (8) can be expressed?

Case (a):

Let K be the conveyance of the channel section at depth of flow y. Let K_o be the conveyance of the channel section at normal depth of flow y_o. From Manning's formula,

$$Q = \frac{1}{n} A R^{2/3} S_f^{\ 1/2} = K S_f^{\ 1/2}$$

where $K = \frac{1}{n} A R^{2/3}$

Hence, by assumption (2) of GVF,

$$K = \frac{Q}{S_f^{1/2}} = \frac{Q}{\sqrt{S_f}} \qquad (9)$$

When the flow is uniform,

$$K_o = \frac{Q}{\sqrt{S_o}} \qquad (10)$$

Dividing (10) by (9),

$$\frac{K_o}{K} = \frac{\left(\frac{Q}{\sqrt{S_o}}\right)}{\left(\frac{Q}{\sqrt{S_f}}\right)} = \frac{\sqrt{S_f}}{\sqrt{S_o}}$$

$$\Rightarrow \frac{K_o^2}{K^2} = \frac{S_f}{S_o} \qquad (11)$$

Let Z be the section factor at depth of flow y.

$$Z = A\sqrt{\frac{A}{T}}$$

$$\Rightarrow Z^2 = \frac{A^3}{T} \qquad (12)$$

Let Z_c be the section factor at critical depth of flow y_c. Then

$$Z_c^{\ 2} = \frac{A_c^3}{T_c} = \frac{Q^2}{g} \qquad (13)$$

Dividing (13) by (12),

$$\frac{Z_c^{\ 2}}{Z^2} = \frac{\left(\frac{Q^2}{g}\right)}{\left(\frac{A^3}{T}\right)} = \frac{Q^2 T}{gA^3} \tag{14}$$

Expressing (7) as

$$\frac{dy}{dx} = \frac{S_o\left(1 - \frac{S_f}{S_o}\right)}{\left(1 - \frac{Q^2 T}{gA^3}\right)}$$

Putting $\left(\frac{S_f}{S_o}\right) = \frac{K_o^2}{K^2}$ from (11) and $\frac{Q^2 T}{gA^3} = \frac{Z_c^{\ 2}}{Z^2}$ from (14) in the above expression,

$$\frac{dy}{dx} = \frac{S_o\left(1 - \frac{K_o^2}{K^2}\right)}{\left(1 - \frac{Z_c^2}{Z^2}\right)} \quad = \quad S_0 \frac{1 - \left(\frac{K_o}{K}\right)^2}{1 - \left(\frac{Z_c}{Z}\right)^2} \tag{15}$$

Equation (15) is useful in developing direct integration techniques.

Case (b):

Let Q_n and Q_c represent the normal discharge and critical discharge at a depth of flow y. Then,

$$Q_n = K\sqrt{S_o} \tag{16}$$

$$\Rightarrow K = \frac{Q}{\sqrt{S_o}} \tag{16 a}$$

$$Q_c = Z\sqrt{g} \tag{17}$$

$$\Rightarrow Z = \frac{Q_c}{\sqrt{g}} \tag{17 a}$$

Substituting (10), (13), (16 a) and (17 a) in (15),

$$\frac{dy}{dx} = S_0 \frac{1-\left(\dfrac{K_o}{K}\right)^2}{1-\left(\dfrac{Z_c}{Z}\right)^2} = S_0 \frac{1-\left\{\dfrac{Q/\sqrt{S_0}}{Q_n/\sqrt{S_0}}\right\}^2}{1-\left\{\dfrac{Q/\sqrt{g}}{Q_c/\sqrt{g}}\right\}^2}$$

$$\Rightarrow \frac{dy}{dx} = S_0 \left\{\frac{1-\left(\dfrac{Q}{Q_n}\right)^2}{1-\left(\dfrac{Q}{Q_c}\right)^2}\right\} \qquad (18)$$

Case (c):

Putting $\frac{dH}{dx} = -S_f$ from (4) and putting $\frac{dZ}{dx} = -S_0$ from (5) in (3),

$$-S_f = -S_0 + \frac{dE}{dx}$$

$$\Rightarrow \frac{dE}{dx} = S_0 - S_f \qquad (19)$$

Equation (19) is called the *differential-energy equation of GVF* to distinguish it from the GVF differential equations (7), (15) and (18). Equation (19) is very helpful in developing numerical techniques for the GVF profile computation.

CLASSIFICATION OF GVF PROFILES

In a given channel, for a fixed discharge Q, Manning's roughness coefficient *n* and bed slope S_0, the normal depth of flow y_n and critical depth of flow y_c are also fixed.

There are three possible relations between the normal depth of flow y_n and critical depth of flow y_c. They are:

(i) $y_0 > y_c$
(ii) $y_0 < y_c$
(iii) $y_0 = y_c$

Further there are two cases where y_0 does not exist. They are:

(i) when the channel bed is horizontal, that is when the bed slope S_0 is zero

(ii) when the channel has an adverse slope, that is when the bed slope S_0 is negative.

Based on the above five cases, the channels are classified into 5 categories as indicated in Table below.

Table 1. Classification of Channels

S. No.	Category of Channel	Symbol	Characteristic Condition	Remark
1.	Mild slope	M	$y_0 > y_c$	Sub-critical flow at normal depth
2.	Steep slope	S	$y_c > y_0$	Super-critical flow at normal depth
3.	Critical slope	C	$y_c = y_0$	Critical flow at normal depth
4.	Horizontal bed	H	$S_0 = 0$	Cannot sustain uniform flow
5.	Adverse slope	A	$S_0 < 0$	Cannot sustain uniform flow